Design of Nonrecursive Digital Filters Using the
Exponential Window

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ABSTRACT- In this paper, nonrecursive digital filter design using the exponential window is investigated [1,2]. The exponential
window proposed by the author [3] has computational cost advantage compared to the well-known Kaiser window due to having
no power series expansion in its time domain representation. First, the filter design equations for the exponential window to meet
the given low pass filter specification are established. And then, the comparison with the other 2-parameter windows, namely
Cosh and Kaiser, is discussed. The simulation results show that the filters designed by the exponential window provide
significantly better far end stopband attenuation, which is important for some applications, than the filters designed by the Cosh
and Kaiser windows, while performing worse stopband attenuations near the stopband frequency.

Keywords: Nonrecursive digital filter, window function, exponential window, cosh window, kaiser window

I. INTRODUCTION

Digital signal processing is an area of electrical engineering and applied mathematics that deals with performing
useful operations on signals in discrete time. Digital filters can be considered as the most important and frequently
used elements in digital signal processing applications. They are classified as finite impulse response (FIR) and
infinite impulse response (IIR) filters by the duration of their impulse response. Each FIR and IIR filters have
advantages and disadvantages. Therefore, neither of them can be considered as best for all situations [2].

FIR filters are very popular because they can be designed as always stable and having linear phase. A disadvantage
of FIR filters over IIR filters is their implementation complexity in case the filter order is very large. The
implementation of FIR filters with nonrecursive techniques guarantees stability [4].

To design nonrecursive FIR digital filters, there are many methods in literature such as optimization methods,
numerical methods, discrete Fourier transform and Fourier series method [5]. Although optimum designs can be
obtained by using the optimization methods such as Remez exchange algorithm, a large amount of computation is
required and this makes the optimization methods unsuitable for real time applications [5]. On the other hand,
Fourier series method with windowing is the most straightforward technique to design nonrecursive filters and
involves a minimal amount of computation compared to other methods. The aim to use a window function (or
window for short) in Fourier series method is to truncate and smooth the infinite duration impulse response of the
ideal filter.

In literature, many windows have been proposed [6-10]. They are classified as fixed or adjustable according to
having number of independent parameters in their functions. Kaiser window [6] is one of the well-known and most
used windows in filter design applications. It is an adjustable window with two independent parameters.

In this paper, performance of the exponential window in nonrecursive digital filter design application is studied. In
Section II, a brief explanation about windows and their use in filter design is given. In Section III, using curve fitting
method, the filter design equations for the exponential window are established. Then, the filters designed by the
exponential, cosh and Kaiser windows are compared in terms of the minimum and maximum stopband attenuation
parameters in Section IV. And the last section concludes the paper.
II. FILTER DESIGN USING WINDOWS

a. Windows

A window, \( w(nT) \), with an odd length of \( N \) is a time domain function which is nonzero for \( |n| \leq (N-1)/2 \) and zero for otherwise. In literature numerous windows exist. They are used to reduce Gibbs’ oscillations resulting from the truncation of Fourier series. The Kaiser window, which is one of the well-known and most used windows in filter design applications, is defined by [6]

\[
 w_k(n) = \begin{cases} 
 I_0(\alpha_k \left[ 1 - \left( \frac{2n}{N-1} \right)^2 \right]) & |n| \leq \frac{N-1}{2} \\
 I_0(\alpha_k) & \text{otherwise}
\end{cases}
\]

where \( \alpha_k \) is the adjustable parameter, and \( I_0(x) \) is the modified Bessel function of the first kind of order zero which can be described by the power series expansion as

\[
 I_0(x) = 1 + \sum_{k=1}^{\infty} \left[ \frac{1}{k! \left( \frac{1}{2} \right)^k} \right]^2
\]

b. Filter design using a window

The impulse response of a realizable noncasual filter using a window, \( w(nT) \), is obtained as

\[
h_{id}(nT) = w(nT)h_{id}(nT)
\]

where \( h_{id}(nT) \) is the infinite duration impulse response of the ideal filter. For a low pass filter with a cut off frequency \( (w_c) \) and sampling frequency \( (w_s=2\pi/T) \), it is [5]

\[
h_{id}(nT) = \begin{cases} 
 \frac{w_c T / \pi}{\sin(w_c nT / n\pi)} & \text{for } n = 0 \\
 \sin(w_c nT / n\pi) & \text{for } |n| \leq w_c / 2
\end{cases}
\]

Throughout this paper the normalized sampling period, \( T=1 \), is considered.

III. FILTER DESIGN USING THE EXPONENTIAL WINDOW

a. Exponential window

The exponential window was proposed by the author of this paper as [1, 3]

\[
w_e(n) = \begin{cases} 
 \exp \left( \alpha_e \left[ 1 - \left( \frac{2n}{N-1} \right)^2 \right] \right) & |n| \leq \frac{N-1}{2} \\
 \exp(\alpha_e) & \text{otherwise}
\end{cases}
\]
The exponential window has two independent parameters, namely the window length \((N)\) and the adjustable shape parameter \((\alpha_e)\). For larger values of \(\alpha_e\), the exponential window becomes to have a Gaussian shape.

b. Filter design equations

To obtain the filter design equations for the exponential window, it is necessary to find the relations between the window parameters and filter spectral parameters. Figure 1 shows the relation between \(\alpha_e\) and the minimum stopband attenuation for \(N = 51\) and \(127\).

Using the curve fitting method in MATLAB, a design equation for the adjustable parameter \((\alpha_e)\) in terms of the minimum stopband attenuation can be obtained as

\[
\alpha_e, \text{Appr} = \begin{cases} 
0 & A_s > 20.8 \\
4.053 \times 10^{-6} A_s^3 - 1.11 \times 10^{-3} A_s^2 + 0.2161 A_s - 4.047 & 20.8 \leq A_s \leq 120
\end{cases}
\]

(6)

Figure 1. Relation between \(\alpha_e\) and \(A_s\) for the exponential window with \(N = 51\) and \(127\)
The approximation model for the adjustable shape parameter given by Eq. (6) is plotted in Figure 2. It is seen that the proposed model provides a good approximation for N = 127. Also, the approximation error for N = 127 is plotted in Figure 3. The largest deviation in alpha is lower than 0.03 which results in a very small error in stopband attenuation.

As a second filter design equation, the relation between the minimum stopband attenuation ($A_s$) and the normalized transition width ($D_t = \Delta f(N-1)/\omega_s$) is required to find the minimum length of the filter which satisfies the given filter specifications. The relation between $D_t$ and $A_s$ for the exponential window is found for N = 51 and 127 empirically and plotted in Figure 4.
By using the curve fitting method, an approximate design relationship between the normalized transition width ($D_f$) and the minimum stopband attenuation ($A_s$) can be established as

$$D_{f,\, \text{Appr}} = \begin{cases} 
0 & A_s < 20.8 \\
9.738 \times 10^{-5} A_s^2 + 67.94 \times 10^{-3} A_s - 0.4784 & 20.8 < A_s \leq 50 \\
72.91 \times 10^{-3} A_s - 0.4769 & 50 < A_s \leq 120 
\end{cases}$$

(7)

The approximation model for the normalized transition width given by Eq. (7) is plotted in Figure 5.

It is seen that the proposed model provides a good approximation for $N = 127$. The relative error of approximated normalized transition width in percent versus the minimum stopband attenuation for $N = 127$ is plotted in Figure 6.
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The percentage error in the model changes between 0.65 and -0.30. This error range satisfies the error criterion in [5] which states that the predicted error in the normalized width must be smaller than 1%.

By using Eq. (7), the minimum odd integer filter length required for satisfying a given minimum stopband attenuation ($A_s$) and transition width ($\Delta w$) can be determined from [8]

$$N \geq \frac{D_f, App \cdot w_s}{\Delta w} + 1 \quad (8)$$

As a result, using the filter design equations given by Eq. (6) through (8), an exponential window can be designed to satisfy the prescribed filter characteristic given in terms of $A_s$ and $\Delta w$.

![Figure 6](image6.png)

Figure 6. Relative error of approximated $D_f$ for the exponential window in percent versus $A_s$ with $N = 127$

![Figure 7](image7.png)

Figure 7. Relation between $D_f$ and $A_{ms}$ for the exponential window with $N = 51$ and 127

Figure 7 shows the effect of the filter length on the relation between the maximum stopband attenuation ($A_{ms}$) and the normalized transition width for the filters designed by the exponential window for $N = 51$ and 127. As it is shown from the figure, an increase in the filter length results in a larger maximum stopband attenuation.

IV. FILTER SPECTRUM COMPARISONS WITH COSH AND KAISER WINDOWS
Figure 8 shows the comparison of the filters designed by the exponential, cosh and Kaiser windows in terms of the minimum stopband attenuation versus the normalized transition width for $N = 127$. It is observed that the filters designed by the Kaiser window perform better minimum stopband attenuation than the filters designed by the exponential window for the same filter length and transition width. As for comparison with the cosh window, the filters designed by the exponential and cosh windows have nearly the same characteristics.

![Figure 8. Minimum stopband attenuation comparison of the filters designed by the exponential, cosh and Kaiser windows for $N = 127$.](image)

For the sake of another comparison with the cosh and Kaiser windows, the far end stopband attenuation, which also gives the maximum stopband attenuation ($A_{\text{max}}$) for the filters designed by the exponential window, is taken as a figure of merit. The attenuation of the far end in stopband is important for some applications [9]. The comparison result is shown in Figure 9. It is seen that as the transition width becomes wider, the filters designed by the exponential window performs better far end suppression than the filters designed by the Kaiser window. As for comparison with the cosh window, the filters designed by the exponential window provides better far end suppression characteristics than the filters designed by the cosh window for $D_t < 5.5$.
Figure 9. Maximum stopband attenuation comparison of the filters designed by the exponential, cosh and Kaiser windows for N = 127

V. CONCLUSION

In this paper, nonrecursive digital filters design using the exponential window has been investigated. The exponential window is a two parameter window function, and it has computational cost advantage compared to the Kaiser window due to having no power series expansion in its time domain representation. First, using the curve fitting method in MATLAB the filter design equations for the exponential window to meet the given low pass filter specification have been established. Then, to test the performance of the exponential window in filter applications, the filters designed by the exponential, cosh and Kaiser windows have been compared. The simulation results showed that the filters designed by the exponential window provide significantly better far end stopband attenuation than the filters designed by the cosh and Kaiser windows, while performing worse stopband attenuations than the Kaiser window near the stopband frequency.

The formalism in this paper reports on odd-length and symmetry case, because only this type of consideration has no restriction on the realizable filter characteristics [10]. But the results can also be generalized to even length case-which allows designing only low pass and band pass filters, and Hilbert transformers [8].

VI. REFERENCES

